

AD-A129 171

EXISTENCE OF SOLUTIONS TO THE NONHOMOGENEOUS STEADY
NAVIER-STOKES EQUATIONS(U) WISCONSIN UNIV-MADISON
MATHEMATICS RESEARCH CENTER C J AMICK APR 83

1/1

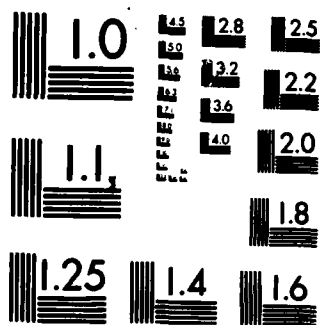
UNCLASSIFIED

MRC-TSR-2509 DAAG29-80-C-0041

F/G 20/4 . NL



END
DATE
FILMED
FBI - SAC
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

MRC Technical Summary Report #2509

EXISTENCE OF SOLUTIONS
TO THE NONHOMOGENEOUS
STEADY NAVIER-STOKES EQUATIONS

AD A129171

Charles J. Amick

Mathematics Research Center
University of Wisconsin-Madison
610 Walnut Street
Madison, Wisconsin 53706

April 1983

(Received March 28, 1983)

DTIC
ELECTE
S JUN 9 1983 D
A

DTIC FILE COPY

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

88 06 07 072

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

EXISTENCE OF SOLUTIONS TO THE NONHOMOGENEOUS
STEADY NAVIER-STOKES EQUATIONS

Charles J. Amick

Technical Summary Report #2509

April 1983

ABSTRACT

This paper concerns the existence of steady solutions to the Navier-Stokes equations in a bounded domain $\Omega \subset \mathbb{R}^2$. The condition of solenoidality for the velocity field imposes a necessary condition on the boundary data. For a certain class of symmetrical domains, ^{the authors} we show that this necessary condition implies the existence of a solution to the problem. The method consists of proving a priori bounds on solutions by assuming the contrary, rescaling the equations, and then arriving at a solution to the steady Euler equations in the limit. Examination of this equation leads to the desired contradiction. After one has suitable bounds on any solutions, one uses the Leray-Schauder theorem to prove existence.

In addition, ^{the authors} we remark on the problem of a general bounded domain $\Omega \subset \mathbb{R}^2$ and suggest how certain maximum principles might yield the expected results.

AMS (MOS) Subject Classifications: 35Q10, 76D05, 35B50

Key Words: Navier-Stokes equations, a priori bounds, maximum principles

Work Unit Number 1 - Applied Analysis

Department of Mathematics, University of Chicago, Chicago, IL.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.



Accession For	
NTIS GRA&I	<input checked="" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
Distribution/	
Availability Codes	
Avail and/or	Special

A

SIGNIFICANCE AND EXPLANATION

The steady Navier-Stokes equations model the flow of a viscous, incompressible fluid in some region Ω . With u and p denoting the velocity and pressure, respectively, one has

$$\left. \begin{aligned} -\nu \Delta u + (u \cdot \nabla) u &= f - \nabla p \\ \nabla \cdot u &= 0 \end{aligned} \right\} \text{ in } \Omega . \quad (*)$$

$$u = \tilde{g} \text{ on } \partial\Omega ,$$

and the problem is to solve for (u, p) for prescribed constant viscosity $\nu > 0$, external force f , and boundary data \tilde{g} . The boundary data is not arbitrary but satisfies the necessary condition

$$0 = \int_{\Omega} \nabla \cdot u = \sum_{i=1}^m \int_{\Gamma_i} \tilde{g}_i \cdot n_i \quad (**)$$

where Γ_i denote the components of $\partial\Omega$ and n_i denotes the outward normal. Condition (**) merely states that the total outflow across $\partial\Omega$ is zero. If one imposes the stronger condition of no outflow across each boundary component:

$$\int_{\Gamma_i} \tilde{g}_i \cdot n_i = 0, \quad i = 1, \dots, m, \quad (***)$$

then classical theory going back to Leray fifty years ago ensures that (*) has a solution for any $\nu > 0$ and any suitably smooth function f .

If (***) is replaced by the necessary condition (**), the standard methods only give the existence of a solution when ν is sufficiently large. In this paper, we show that the necessary condition on \tilde{g} yields a solution (u, p) to (*) in a certain class of domains Ω in the plane. We also remark on the case of a general domain in the plane, and show how certain maximum principles are applicable.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

EXISTENCE OF SOLUTIONS TO THE NONHOMOGENEOUS STEADY NAVIER-STOKES EQUATIONS

Charles J. Amick

1. INTRODUCTION

In this paper we consider the steady Navier-Stokes equations in a bounded domain (an open, connected set) $\Omega \subset \mathbb{R}^2$:

$$\left. \begin{aligned} -v_0 \Delta u + (u \cdot \nabla)u &= f - \nabla p, \\ \nabla \cdot u &= 0, \end{aligned} \right\} \text{ in } \Omega \quad (1.1)$$

$$(1.2)$$

$$u = \tilde{g} \text{ on } \partial\Omega. \quad (1.3)$$

We assume that $\partial\Omega$ is infinitely differentiable, $f \in C^\infty(\bar{\Omega} \times \mathbb{R}^2)$, $\tilde{g} \in C^\infty(\partial\Omega \times \mathbb{R}^2)$, and $v_0 > 0$. The problem is to find a velocity field $u = (u_1, u_2)$ and a pressure p satisfying (1.1) - (1.3). The boundary data \tilde{g} is not arbitrary, but satisfies the compatibility condition

$$0 = \int_{\Omega} \operatorname{div} u = \sum_{i=1}^m \int_{\Gamma_i} \tilde{g}_i \cdot n_i, \quad (1.4)$$

where the Γ_i are the components of $\partial\Omega$ and n_i denotes the outward normal to Γ_i . We shall assume throughout this paper that (1.4) holds. Our intention is to prove the existence of a solution (u, p) to (1.1) - (1.3) for any \tilde{g} satisfying the necessary condition (1.4). The results will hold for a certain class of domains $\Omega \subset \mathbb{R}^2$, and we shall remark on the general case in section 3.

If one replaces (1.4) by the stronger condition

$$\int_{\Gamma_i} \tilde{g}_i \cdot n_i = 0, \quad i = 1, \dots, m, \quad (1.5)$$

then the existence of a solution (u, p) for any $v_0 > 0$ is classical (at

Department of Mathematics, University of Chicago, Chicago, IL.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

least for domains $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3) and goes back to the fundamental work of Leray [9] (see also [2], [8], [12]). Condition (1.5) has been assumed in the recent work of Foias and Temam [3], [4], Saut and Temam [10], and Temam [11], [12] on properties of solutions to the steady Navier-Stokes equations. As we shall see in section 2, the main advantage of (1.5) is that it allows the boundary data \hat{g} to be extended to Ω as a curl; more precisely, there is a smooth function ψ defined on $\bar{\Omega}$ such that $\nabla \times \psi = \hat{g}$ on $\partial\Omega$. One then mollifies ψ near to $\partial\Omega$ and obtains a priori bounds suitable for the Leray-Schauder theorem.

If (1.4) holds but not (1.5), then one cannot extend \hat{g} as a curl to Ω . However, standard theory [8], [12] ensures the existence of a function $g \in C^\infty(\bar{\Omega} \rightarrow \mathbb{R}^2)$ satisfying the Stokes equations:

$$\left. \begin{aligned} -\nu \Delta g &= -\nabla q \\ \nabla \cdot g &= 0 \end{aligned} \right\} \text{ in } \Omega \quad (1.6)$$

$$(1.7)$$

$$g = \hat{g} \text{ on } \partial\Omega. \quad (1.8)$$

If we set $u = g + v$, then (1.1) - (1.3) is equivalent to solving the problem

$$\left. \begin{aligned} -\nu \Delta v + (g \cdot \nabla)v + (v \cdot \nabla)v + (v \cdot \nabla)g &= \hat{f} - \nabla p \\ \nabla \cdot v &= 0 \end{aligned} \right\} \text{ in } \Omega \quad (1.9)$$

$$v = 0 \text{ on } \partial\Omega, \quad (1.10)$$

$$(1.11)$$

where $\hat{f} = f - (g \cdot \nabla)g$ is fixed and we absorbed the q term into p . Let $H(\Omega)$ denote the completion of $\{v = (v_1, v_2) \in C_0^\infty(\Omega \rightarrow \mathbb{R}^2) : \operatorname{div} v = 0 \text{ in } \Omega\}$ in the Dirichlet norm:

$$\langle v, v \rangle_H = |v|_H^2 = \sum_{j=1}^2 \int_{\Omega} \left| \frac{\partial v_j}{\partial x_j} \right|^2.$$

The problem of solving (1.9) - (1.11) for (v, p) is equivalent to finding a (weak) solution v of the following equation:

$$\begin{aligned} v_0 \langle \phi, v \rangle_H + \int_{\Omega} \phi \cdot (g \cdot \nabla) v + \int_{\Omega} \phi \cdot (v \cdot \nabla) v + \int_{\Omega} \phi \cdot (v \cdot \nabla) g \\ = \int_{\Omega} \phi \cdot \tilde{f} \quad \text{for all } \phi \in H(\Omega) \end{aligned} \quad (1.12)$$

In the usual way [8], [12], this is equivalent to solving an operator equation

$$\dot{v}_0 v = Av + F, \quad (1.13)$$

where $F \in H(\Omega)$ and A is a compact map of $H(\Omega)$ into itself. Here F and A are defined by the Riesz representation theorem:

$$\int_{\Omega} \phi \cdot (g \cdot \nabla) v + \int_{\Omega} \phi \cdot (v \cdot \nabla) v + \int_{\Omega} \phi \cdot (v \cdot \nabla) g = -\langle \phi, Av \rangle_H$$

and

$$\int_{\Omega} \phi \cdot \tilde{f} = \langle \phi, F \rangle_H$$

for all $\phi \in H(\Omega)$. After one has found a solution $v \in H$ to (1.13), it is then standard to recover a smooth solution (p, v) of (1.9) - (1.11).

For each $v > v_0$, let

$$S(v) = \{v \in H(\Omega) : vv = Av + F\}. \quad (1.14)$$

Note that $v \in S(v)$ if and only if there is a pressure p such that $(g+v, p)$ satisfy (1.1) - (1.3) with v_0 replaced by v . We shall prove in Theorem 2.1 that $S(v)$ is non-empty if v exceeds some critical value \tilde{v} and that

$$|v|_H < \text{const.}/v, \quad v > \tilde{v}. \quad (1.15)$$

Our main result in section 2 is the following estimate:

$$\sup_{v \in [v_0, \tilde{v}]} \sup_{v \in S(v)} |v|_H < \infty. \quad (1.16)$$

This ensures that all solutions of the equation

$$v = \frac{\lambda}{v_0} (Av + F), \quad (1.17)$$

$\lambda \in [0, 1]$, are uniformly bounded in $H(\Omega)$, and so the existence of a solution to (1.13) follows by the Leray-Schauder theorem. We shall prove (1.16) by assuming it is false, and then deriving a contradiction. Although (1.15) holds for bounded domains in \mathbb{R}^2 or \mathbb{R}^3 , we have only succeeded in proving (1.16) for a certain class of domains $\Omega \subset \mathbb{R}^2$ and data \tilde{g} and f .

Since (1.4) and (1.5) are the same for $m = 1$, we shall restrict attention at all times to domains with $m \geq 2$ boundary components.

DEFINITION: A bounded domain $\Omega \subset \mathbb{R}^2$ is said to be admissible if
 (a) $\partial\Omega$ is of class C^∞ , (b) $\partial\Omega$ consists of $m \geq 2$ components Γ_i , (c) Ω is symmetric about the line $\{x_2 = 0\}$ and (d) each component Γ_i intersects the line $\{x_2 = 0\}$.

A function $h = (h_1, h_2)$ mapping Ω or $\partial\Omega$ into \mathbb{R}^2 is said to be symmetric about the line $\{x_2 = 0\}$ if h_1 is an even function of x_2 while h_2 is an odd function of x_2 .

DEFINITION: A pair (f, \tilde{g}) is said to be admissible data if (a)
 $f \in C^\infty(\bar{\Omega} \rightarrow \mathbb{R}^2)$, (b) $\tilde{g} \in C^\infty(\partial\Omega \rightarrow \mathbb{R}^2)$, and (c) f and \tilde{g} are symmetric about the line $\{x_2 = 0\}$.

If Ω is an admissible domain and (f, \tilde{g}) is admissible data, it is natural to seek a solution (u, p) to (1.1) - (1.3) with u symmetric about $\{x_2 = 0\}$; the corresponding pressure p will be an even function of x_2 . The main result of this paper is the following theorem which is proved in section 2.

THEOREM 1.1. Let $\Omega \subset \mathbb{R}^2$ be an admissible domain and let (f, \tilde{g}) be admissible data. Then for every $\nu_0 > 0$ there exists a solution
 $(u, p) \in C^\infty(\bar{\Omega} \rightarrow \mathbb{R}^2) \times C^\infty(\bar{\Omega} \rightarrow \mathbb{R})$ of (1.1) - (1.3). The function u is symmetric about $\{x_2 = 0\}$ and the pressure p is an even function of x_2 .

Acknowledgement. I am indebted to Professor J. Heywood for bringing this problem to my attention. In addition, he strongly suggested that the proof of (1.16) might follow by assuming the contrary and deriving a contradiction.

2. A PRIORI BOUNDS

Before proceeding to the proof of Theorem 1.1, we show how the condition (1.5) leads to solutions of (1.1) - (1.3) for any $v_0 > 0$. Assume for the moment that Ω is a bounded domain in \mathbb{R}^n ($n = 2$ or 3) with smooth boundary and f and \hat{g} are smooth on $\bar{\Omega}$ and $\partial\Omega$, respectively. Equation (1.5) allows one to extend the boundary data \hat{g} to Ω as a curl; $g = \nabla \times \psi$ in $\bar{\Omega}$ and $g = \hat{g}$ on $\partial\Omega$ [2], [8], [12]. For each $\varepsilon > 0$, let $\rho(\cdot; \varepsilon)$ denote a suitable mollifier [2; p. 209], [8; p. 108], [12; p. 175] with $\rho \equiv 1$ near to $\partial\Omega$ and with support in an ε -neighborhood of $\partial\Omega$. Upon choosing ε sufficiently small and setting $g_\varepsilon = \nabla \times (\rho(\cdot; \varepsilon) \psi(\cdot))$, one has [2; p. 210], [8; p. 109], [12; p. 175]

$$\left| \int_{\Omega} g_\varepsilon \cdot (\nabla \cdot \nabla) v \right| < \text{const. } \varepsilon |v|_H^2 \text{ for all } v \in H(\Omega) \quad (2.1)$$

and the constant is independent of ε and v . One seeks a solution of (1.1) - (1.3) of the form $u = g_\varepsilon + v$, so that v is to satisfy (1.9) - (1.11) (with g replaced by g_ε and \hat{f} replaced by $\hat{f} + v_0 \Delta g_\varepsilon$), or, equivalently, (1.13). With $S(v)$ as in (1.14), it suffices to choose ε such that $\sup_{v > v_0} \sup_{v \in S(v)} |v|_H < \infty$. If $v \in S(v)$, then setting $\phi = v$ in (1.12) yields

$$\begin{aligned} v |v|_H^2 &= - \int_{\Omega} v \cdot (\nabla \cdot \nabla) g_\varepsilon + \int_{\Omega} v \cdot \hat{f} \\ &= \int_{\Omega} g_\varepsilon \cdot (\nabla \cdot \nabla) v + \int_{\Omega} v \cdot \hat{f} \\ &< \text{const. } \varepsilon |v|_H^2 + \frac{\text{const.}}{\varepsilon} |\hat{f}|_{L^2(\Omega)}^2. \end{aligned} \quad (2.2)$$

If we choose ε_0 such that $\text{const. } \varepsilon_0 < v_0/2$, then

$$|v|_H^2 < \frac{\text{const.}}{\varepsilon_0 v_0} |\hat{f}|_{L^2(\Omega)}^2 \quad (2.3)$$

for all $v \in S(v)$ if $v > v_0$. It then follows from the Leray-Schauder theorem that a solution to (1.13) exists.

The key point in the analysis above was the representation $g = \nabla \times \psi$ which allowed us to put a mollifier with small support near $\partial\Omega$ inside the curl. The form of g_ϵ led to (2.1) which was the key step in proving (2.3) from (2.2). If one drops the condition (1.5) (but still has the necessary condition (1.4)), then it is not clear how to proceed. One might set $M = \{g \in C^\infty(\bar{\Omega} \rightarrow \mathbb{R}^n) : \nabla \cdot g = 0 \text{ in } \Omega \text{ and } g = \tilde{g} \text{ on } \partial\Omega\}$ and set

$$\mu = \inf_{g \in M} \max_{v \in H(\Omega)} \frac{\int_{\Omega} g \cdot (\nabla \cdot \nabla) v}{|v|_H^2}.$$

If $\mu < \nu_0$, then one gets a priori bounds for $v \in S(v)$, $v > \nu_0$, by (2.2). When \tilde{g} satisfies (1.5), then (2.1) gives $\mu = 0$. Unfortunately, in general one will not have $\mu < \nu_0$, and so a different approach must be taken.

The following theorem gives the existence of a unique solution to (1.1) - (1.3) if ν is sufficiently large.

THEOREM 2.1. Let Ω be a bounded domain in \mathbb{R}^n ($n = 2$ or 3) with $\partial\Omega$ of class C^∞ , and let $f \in C^\infty(\bar{\Omega} \rightarrow \mathbb{R}^n)$ and $\tilde{g} \in C^\infty(\partial\Omega \rightarrow \mathbb{R}^n)$. Then there exists $\tilde{\nu} > 0$ such that (1.1) - (1.3) has a solution $(u, p) \in C^\infty(\bar{\Omega} \rightarrow \mathbb{R}^n) \times C^\infty(\bar{\Omega} \rightarrow \mathbb{R})$ for all $\nu > \tilde{\nu}$. In addition, (u, p) is unique up to an additive constant for p .

Proof. Let g be as in (1.6) - (1.8). Now

$$\left| \int_{\Omega} g \cdot (\nabla \cdot \nabla) v \right| \leq \text{const.} \|g\|_{L^\infty(\Omega)} |v|_H^2 = C |v|_H^2, \quad v \in H(\Omega),$$

and the constant C is independent of ν . Let $\tilde{\nu}$ be any fixed number greater than $2C$. We claim that (1.1) - (1.3) has a solution for any $\nu > \tilde{\nu}$. Indeed, if ν_0 is so given, then it suffices to solve (1.13). If $v \in S(\nu)$, then (2.2) gives

$$\nu |v|_H^2 = \int_{\Omega} g \cdot (\nabla \cdot \nabla) v + \int_{\Omega} \tilde{f} v \leq C |v|_H^2 + \text{const} \|\tilde{f}\|_{L^2(\Omega)} |v|_H,$$

whence

$$|v|_H \leq \frac{\text{const.} \cdot |\tilde{f}|}{v} L^2(\Omega) \quad (2.4)$$

for all $v > v_0$. This estimate gives an a priori bound for solutions to (1.17), and so existence follows from the Leray-Schauder theorem.

If (u, p) and (w, q) satisfy (1.1) - (1.3), then a calculation yields

$$\begin{aligned} v_0 |u-w|_H^2 &= \int_{\Omega} w \cdot ((u-w) \cdot \nabla) (u-w) \\ &\leq \text{const.} |w|_{L^4(\Omega)} |u-w|_{L^4(\Omega)} |u-w|_H \\ &\leq \text{const.} |u-w|_H^2 \end{aligned}$$

by (2.4). The constant is independent of v_0 , and so if \tilde{v} is sufficiently large, then $v_0 > \tilde{v}$ ensures a unique solution of (1.13). The existence of a pressure p and the regularity of (u, p) is standard [8], [12], q.e.d.

Throughout the rest of this section, we shall assume that $\Omega \subset \mathbb{R}^2$ is an admissible domain and (f, \tilde{g}) are admissible data. Let g be a solenoidal extension of \tilde{g} to Ω as in (1.6) - (1.8); since \tilde{g} and Ω are symmetric about $\{x_2 = 0\}$, the extension g may be taken to be symmetric. We seek a solution u of (1.1) - (1.3) of the form $u = g + v$, where $v \in H(\Omega)$ and v is to be symmetric about $\{x_2 = 0\}$. Let $H_s(\Omega)$ denote the closed subspace of $H(\Omega)$ consisting of velocity fields which are symmetric about $\{x_2 = 0\}$. It is a Hilbert space with the previous inner product $\langle \cdot, \cdot \rangle_H$. The problem of solving (1.1) - (1.3) is equivalent to finding $v \in H_s(\Omega)$ satisfying $v_0 v = Av + F$, where $F \in H_s(\Omega)$ and A is a compact map of H_s into itself. If we set

$$S(v) = \{v \in H_s(\Omega) : vv = Av + F\},$$

then the proof of Theorem 2.1 ensures that $S(v)$ is non-empty for all $v > \tilde{v}$ and that (1.15) holds. In order to prove (1.16) (and thereby Theorem 1.1), we shall assume the contrary and derive a contradiction.

Let $v_n \in S(v_n)$ with $|v_n|_H \rightarrow \infty$ as $n \rightarrow \infty$. Without loss of generality we may assume that $v_n \rightarrow v \in [v_0, \tilde{v}]$ as $n \rightarrow \infty$. Since $v_n \in S(v_n)$, it satisfies (1.12) with v_0 replaced by v_n . Standard theory gives the existence of a pressure p_n (normalized by $\int_{\Omega} p_n = 0$) such that

$$-v_n \Delta v_n + (g \cdot \nabla) v_n + (v_n \cdot \nabla) g + (v_n \cdot \nabla) v_n = \tilde{f} - \nabla p_n$$

$$= f - (g \cdot \nabla) g - \nabla p_n \quad \text{in } \Omega. \quad (2.5)$$

If we set $u_n = g + v_n$, then

$$-v_n \Delta u_n + (u_n \cdot \nabla) u_n = f - \nabla p_n \quad \text{in } \Omega$$

and we have absorbed the term $-v_n \Delta g \equiv \frac{-v_n}{v_0} \nabla g$ into the p_n term. Define

$$\tilde{u}_n = u_n / |v_n|_H, \quad \tilde{v}_n = v_n / |v_n|_H \quad \text{and} \quad \tilde{p}_n = p_n / |v_n|_H^2.$$

Without loss of generality, we may assume that $\tilde{v}_n \rightarrow \tilde{v}$ in $H_s(\Omega)$ and $\tilde{v}_n \rightarrow \tilde{v}$ in $L^q(\Omega)$, $q \in [1, \infty)$. The estimates of Cattabriga [1], [12] applied to (2.5) yield

$$|p_n|_{W^{1,q}(\Omega)} \leq \text{const.} \{ |\tilde{f}|_{L^q(\Omega)} + |(g \cdot \nabla) v_n + (v_n \cdot \nabla) v_n + (v_n \cdot \nabla) g|_{L^q(\Omega)} \}$$

$$\leq \text{const.} \{ 1 + |v_n|_H^2 \}$$

for any $q \in (1, 2)$. In particular, \tilde{p}_n is bounded in $W^{1,q}(\Omega)$, $q \in (1, 2)$, independently of n , and so $\tilde{p}_n \rightarrow \tilde{p}$ in $W^{1,q}(\Omega)$, $q \in (1, 2)$, and $\tilde{p}_n \rightarrow \tilde{p}$ in $L^q(\Omega)$, $q \in [1, \infty)$. If we multiply (2.5) by v_n , integrate over Ω , and then divide by $|v_n|_H^2$, there results

$$v_n = \int_{\Omega} g \cdot (\tilde{v}_n \cdot \nabla) \tilde{v}_n + \frac{1}{|v_n|_H} \int_{\Omega} \tilde{f} \cdot \tilde{v}_n. \quad (2.6)$$

Since $\tilde{v}_n \rightarrow \tilde{v}$ in $L^2(\Omega)$, we may take the limit of $n \rightarrow \infty$ in (2.6):

$$v = \int_{\Omega} g \cdot (\tilde{v} \cdot \nabla) \tilde{v}. \quad (2.7)$$

If we multiply (2.5) by a fixed $\phi \in C_0^\infty(\Omega \rightarrow \mathbb{R}^2)$, integrate over Ω , divide by $|v_n|_H^2$ and let $n \rightarrow \infty$, there results

$$\int_{\Omega} \phi \cdot (\tilde{v} \cdot \nabla) \tilde{v} = - \int_{\Omega} \phi \cdot \nabla \tilde{p}, \quad \phi \in C_0^\infty(\Omega \rightarrow \mathbb{R}^2). \quad (2.8)$$

Since ϕ was arbitrary, it follows that (\hat{v}, \hat{p}) , with $\hat{v} \in H_g(\Omega)$, is a weak solution of the steady Euler equations;

$$(\hat{v} \cdot \nabla) \hat{v} = -\nabla \hat{p} \text{ almost everywhere in } \Omega. \quad (2.9)$$

Recall that the solenoidality of \hat{v} is immediate from its membership in $H_g(\Omega)$ while $\hat{v} = 0$ on $\partial\Omega$ (in the sense of a trace) for the same reason. Since $\hat{v} \in H_g(\Omega)$ and $\hat{p} \in W^{1,q}(\Omega)$, $q \in (1,2)$, it follows that $(\hat{v} \cdot \nabla) \hat{v}$, $\nabla \hat{p} \in L^q(\Omega)$, $q \in (1,2)$, and so (2.8) actually holds for all $\phi \in L^r(\Omega)$, $r \in (2, \infty)$. In particular, setting $\phi = g$ yields

$$\begin{aligned} \int_{\Omega} g (\hat{v} \cdot \nabla) \hat{v} &= - \int_{\Omega} g \nabla \hat{p} = - \int_{\Omega} \operatorname{div}(g \hat{p}) \\ &= \int_{\partial\Omega} \hat{p} g \cdot n = \sum_{i=1}^m \int_{\Gamma_i} \hat{p} \hat{g}_i \cdot n_i. \end{aligned} \quad (2.10)$$

The integration by parts in (2.9) is justified since $\hat{p} \in W^{1,q}(\Omega)$ has a well-defined trace on $\partial\Omega$.

To motivate what is about to follow, we argue formally for the moment. Since $\hat{v} \in H_g(\Omega)$, we have $\hat{v} = 0$ on $\partial\Omega$ in the sense of a trace, and so (2.9) suggests that \hat{p} is a constant C_i on each component Γ_i of $\partial\Omega$. If we could show that these constants are all equal to, say, C , then (2.10) would give

$$\int_{\Omega} g (\hat{v} \cdot \nabla) \hat{v} = C \sum_{i=1}^m \int_{\Gamma_i} \hat{g}_i \cdot n_i = 0 \quad (2.11)$$

by (1.4). However, (2.7) shows that (2.10) is impossible, and so we will have the desired contradiction. In the following two theorems, we justify these formal arguments.

THEOREM 2.2. The trace of \hat{p} on Γ_i is a constant C_i , $i = 1, \dots, m$, almost everywhere on Γ_i .

Proof. Let i be arbitrary and set $\Gamma = \Gamma_i$. Let $z_0 \in \Gamma$ and change to a new orthogonal coordinate system (\hat{x}_1, \hat{x}_2) centered at z_0 with the \hat{x}_2 -axis pointing along the inner normal to Γ at z_0 . We write

$\tilde{\nabla} = (\frac{\partial}{\partial \tilde{x}_1}, \frac{\partial}{\partial \tilde{x}_2})$. For small $\epsilon > 0$, the boundary component Γ is given locally by $\tilde{x}_2 = h(\tilde{x}_1)$, $\tilde{x}_1 \in (-\epsilon, \epsilon)$, with $h \in C^\infty$. Let $\delta > 0$ be sufficiently small such that

$$A = A(\epsilon, \delta) = \{(\tilde{x}_1, \tilde{x}_2) : \tilde{x}_1 \in (-\epsilon, \epsilon), \tilde{x}_2 \in (h(\tilde{x}_1), h(\tilde{x}_1) + \delta)\} \subset \Omega.$$

Now

$$\begin{aligned} \int_A \frac{|(\tilde{\nabla} \cdot \tilde{\nabla})\tilde{v}|}{|\tilde{x}_2 - h(\tilde{x}_1)|} d\tilde{x}_1 d\tilde{x}_2 &\leq \text{const.} \left(\int_{-\epsilon}^{\epsilon} \int_{h(\tilde{x}_1)}^{h(\tilde{x}_1) + \delta} \frac{|\tilde{v}(\tilde{x}_1, \tilde{x}_2)|^2}{|\tilde{x}_2 - h(\tilde{x}_1)|^2} d\tilde{x}_2 d\tilde{x}_1 \right)^{1/2} |\tilde{v}|_H \\ &\leq \text{const.} (4 \int_{-\epsilon}^{\epsilon} \int_{h(\tilde{x}_1)}^{h(\tilde{x}_1) + \delta} |\tilde{\nabla} \tilde{v}|^2)^{1/2} |\tilde{v}|_H \leq \text{const.} |\tilde{v}|_H^2 = \text{const.}, \end{aligned} \quad (2.12)$$

where we have used the standard estimate

$$\int_a^{a+\delta} \left(\frac{w(\tilde{x}_2)}{\tilde{x}_2 - a} \right)^2 d\tilde{x}_2 \leq 4 \int_a^{a+\delta} (w'(\tilde{x}_2))^2 d\tilde{x}_2 \quad (2.13)$$

for functions $w \in C^1[a, a+\delta]$ which vanish at $\tilde{x}_2 = a$. Since $H(\Omega)$ is the completion of C_0^∞ functions, one first uses (2.13) to prove (2.12) for such functions and then takes the limit.

Since $(\tilde{\nabla} \cdot \tilde{\nabla})\tilde{v} = -\tilde{\nabla} p$, it follows from (2.12) that

$$\int_{A(\epsilon, \delta)} |\tilde{\nabla} p| = o(\delta) \text{ as } \delta \rightarrow 0.$$

If $\phi \in C_0^\infty(-\epsilon, \epsilon)$, then

$$\begin{aligned} \int_{A(\epsilon, \delta)} \phi'(\tilde{x}_1) \tilde{p}(\tilde{x}_1, \tilde{x}_2) d\tilde{x}_1 d\tilde{x}_2 &= \int_{-\epsilon}^{\epsilon} \int_0^\delta \phi'(\tilde{x}_1) \tilde{p}(\tilde{x}_1, h(\tilde{x}_1) + \tilde{x}_2) d\tilde{x}_2 d\tilde{x}_1 \\ &= - \int_{-\epsilon}^{\epsilon} \int_0^\delta \phi(\tilde{x}_1) \left\{ \frac{\partial \tilde{p}}{\partial \tilde{x}_1}(\tilde{x}_1, h(\tilde{x}_1) + \tilde{x}_2) + h'(\tilde{x}_1) \frac{\partial \tilde{p}}{\partial \tilde{x}_2}(\tilde{x}_1, h(\tilde{x}_1) + \tilde{x}_2) \right\} d\tilde{x}_2 d\tilde{x}_1 \\ &= o(\delta) \text{ as } \delta \rightarrow 0. \end{aligned}$$

If we divide both sides of this equation by δ and let $\delta \rightarrow 0$, there results

$$\int_{-\epsilon}^{\epsilon} \phi'(\tilde{x}_1) \tilde{p}(\tilde{x}_1, h(\tilde{x}_1)) d\tilde{x}_1 = 0 \text{ for all } \phi \in C_0^\infty(-\epsilon, \epsilon).$$

It follows that \tilde{p} is a constant on Γ almost everywhere. g.e.d.

Theorem 2.2 did not use the symmetry of Ω , but we shall need it for Theorem 2.3. We now introduce some notation for the boundary components Γ_i . Without loss of generality, we may assume that Γ_1 is the 'exterior' component; that is, $\Omega \subset \text{int } \Gamma_1$. Since Ω is admissible, the set $\{x_2 = 0\} \cap \Gamma_1$ consists of two points $(\alpha_1, 0)$ and $(\beta_1, 0)$ with, say, $\alpha_1 < \beta_1$. We may label the components so that $\alpha_1 < \alpha_2 < \beta_2 \dots < \alpha_m < \beta_m < \beta_1$. Note that the sets $\{(x_1, 0) : \alpha_1 < x_1 < \alpha_2\}$, $\{(x_1, 0) : \beta_1 < x_1 < \alpha_{i+1}\}$, $i = 2, \dots, m-1$, and $\{(x_1, 0) : \beta_m < x_1 < \beta_1\}$ are all contained in Ω .

THEOREM 2.3. The constants C_i in Theorem 2.2 are all equal: $C_i = C$, $i = 1, \dots, m$.

Proof. We shall prove that $C_1 = C_2$ since the other cases are similar. Near to the point $(\alpha_1, 0)$, the component Γ_1 has the form $\{(h_1(x_2), x_2) : x_2 \in (-\delta, \delta)\}$ for some small $\delta > 0$. Here $h_1 \in C^\infty$ and $h_1(0) = \alpha_1$. Similarly, Γ_2 has the form $\{(h_2(x_2), x_2) : x_2 \in (-\delta, \delta)\}$ near to $(\alpha_2, 0)$. Set $A = A(\delta) = \{(x_1, x_2) : x_1 \in (h_1(x_2), h_2(x_2)), x_2 \in (0, \delta)\}$, so that $A(\delta) \subset \Omega$ for all small δ . Define the total-head pressure $\tilde{\phi} = \tilde{p} + \frac{1}{2} |\tilde{v}|^2 = \tilde{p} + \frac{1}{2} (\tilde{v}_1^2 + \tilde{v}_2^2)$, where $\tilde{v} = (\tilde{v}_1, \tilde{v}_2)$. Since $(\tilde{v} \cdot \nabla) \tilde{v} = -\nabla \tilde{p}$ (almost everywhere) in Ω , we have

$$\begin{aligned} \frac{\partial}{\partial x_1} \tilde{\phi} &= \frac{\partial \tilde{p}}{\partial x_1} + \tilde{v}_1 \frac{\partial \tilde{v}_1}{\partial x_1} + \tilde{v}_2 \frac{\partial \tilde{v}_2}{\partial x_1} \\ &= (-\tilde{v}_1 \frac{\partial \tilde{v}_1}{\partial x_1} - \tilde{v}_2 \frac{\partial \tilde{v}_1}{\partial x_2}) + \tilde{v}_1 \frac{\partial \tilde{v}_1}{\partial x_1} + \tilde{v}_2 \frac{\partial \tilde{v}_2}{\partial x_1} \\ &= -\tilde{v}_2 \tilde{\omega} \text{ almost everywhere in } \Omega, \end{aligned} \tag{2.14}$$

where $\tilde{\omega} = \frac{\partial \tilde{v}_1}{\partial x_2} - \frac{\partial \tilde{v}_2}{\partial x_1}$ denotes the vorticity. Integrating this equation over $A(\delta)$ yields

$$\int_0^\delta \{ \tilde{p}(h_2(x_2), x_2) - \tilde{p}(h_1(x_2), x_2) \} dx_2 = \delta(C_2 - C_1) \\ = - \int_{A(\delta)} \tilde{v}_2 \tilde{\omega} ,$$

or, equivalently

$$|C_2 - C_1| = \frac{1}{\delta} \left| \int_{A(\delta)} \tilde{v}_2 \tilde{\omega} \right| . \quad (2.15)$$

If we extend \tilde{v}_2 to $\mathbb{R} \times (0, \delta)$ as zero outside $A(\delta)$, then

$$\int_{A(\delta)} \frac{|\tilde{v}_2(x_1, x_2)|^2}{x_2^2} = \int_{-\infty}^{\infty} \int_0^\delta \frac{|\tilde{v}_2(x_1, x_2)|^2}{x_2^2} dx_2 dx_1 \\ < 4 \int_{-\infty}^{\infty} \int_0^\delta \left| \frac{\partial}{\partial x_2} \tilde{v}_2 \right|^2 < 4 \int_{A(\delta)} |\nabla \tilde{v}|^2$$

where we have used (2.13) and the fact that $\tilde{v}_2(x_1, 0) = 0$ by symmetry. The use of these estimates in (2.15) give

$$|C_1 - C_2|^2 < \frac{1}{\delta^2} \left(\int_{A(\delta)} x_2^2 \frac{|\tilde{v}_2|^2}{x_2^2} \right) \left(\int_{A(\delta)} \tilde{\omega}^2 \right) \\ < \frac{1}{\delta^2} (4\delta^2 \int_{A(\delta)} |\nabla \tilde{v}|^2) (4 \int_{A(\delta)} |\nabla \tilde{v}|^2) = 16 \left(\int_{A(\delta)} |\nabla \tilde{v}|^2 \right)^2 \\ \rightarrow 0 \text{ as } \delta \rightarrow 0 .$$

Hence, $C_1 = C_2$. a.e.d.

As noted before, the use of Theorem 2.3 and (1.4) in (2.10) gives a contradiction and so (1.16) holds. The Leray-Schauder theorem then gives the existence of a solution $v \in H_g(\Omega)$ to (1.13), and the existence of a pressure p and the regularity of (u, p) following in a standard way.

3. REMARKS ON THE GENERAL CASE

We now consider the problem of solving (1.1) - (1.3) in a general bounded domain $\Omega \subset \mathbb{R}^2$. We assume $\partial\Omega$ is smooth and that it consists of $m > 1$ components. Furthermore, f and \tilde{g} are smooth functions from $\bar{\Omega}$ and $\partial\Omega$,

respectively, to R^2 . In order to prove the existence of a solution, it suffices to prove (1.16). If we assume this is false, then there is a sequence with $v_n \in S(v_n)$ and $|v_n|_H \rightarrow \infty$ as $n \rightarrow \infty$. Setting $\tilde{v}_n = v_n/|v_n|_H$ and $\tilde{p}_n = p_n/|v_n|_H^2$ yields $\tilde{v}_n \rightarrow \tilde{v}$ in $H(\Omega)$, $\tilde{v}_n \rightarrow v$ in $L^q(\Omega)$, $q \in [1, \infty)$, $\tilde{p}_n \rightarrow \tilde{p}$ in $W^{1,q}(\Omega)$, $q \in (1, 2)$, and $\tilde{p}_n \rightarrow \tilde{p}$ in $L^q(\Omega)$, $q \in [1, \infty)$. The pair (\tilde{v}, \tilde{p}) is a weak solution of the steady Euler equations $(\tilde{v} \cdot \nabla) \tilde{v} = -\nabla \tilde{p}$ in Ω . Theorem 2.2 shows that \tilde{p} is a constant C_i on each component Γ_i of $\partial\Omega$. In order to derive a contradiction from (2.7), it suffices to show that the C_i are all equal.

Theorem 2.3 showed that if $v \in H_g(\Omega)$ is a weak solution of the steady Euler equations in an admissible domain Ω , then the corresponding pressure \tilde{p} has the same constant value on each of its components. If we could show for general Ω that

$$\tilde{v} \in H(\Omega) \text{ and } (\tilde{v} \cdot \nabla) \tilde{v} = -\nabla \tilde{p} \text{ in } \Omega \text{ implies } \tilde{p} = C \text{ on } \Gamma_i, i = 1, \dots, m, \quad (3.1)$$

then we would have solved (1.1) - (1.3). Unfortunately, (3.1) is not true as the following simple example shows.

EXAMPLE 3.1. Let $\Omega = \{(r, \theta) : 1 < r = \sqrt{x_1^2 + x_2^2} < R, 0 < \theta < 2\pi\}$ and let $\psi \in C^1[1, R]$ with $\psi'(1) = \psi'(R) = 0$ and $\psi'' \in L^2(1, R)$. Define $\tilde{v}(r, \theta) = (\frac{\partial}{\partial x_2} \psi(r), -\frac{\partial}{\partial x_1} \psi(r)) = (\frac{x_2}{r} \psi'(r), -\frac{x_1}{r} \psi'(r)) \in H(\Omega)$ and define $\tilde{p}(r) = \int_1^r (\psi'(w))^2/w \, dw, r \in [1, R]$.

Then (\tilde{v}, \tilde{p}) is a weak solution of the steady Euler equations in Ω :

$$\left. \begin{aligned} (\tilde{v} \cdot \nabla) \tilde{v} &= -\nabla \tilde{p} \\ \nabla \cdot \tilde{v} &= 0 \end{aligned} \right\} \text{ in } \Omega,$$

$$\tilde{v} = 0 \text{ on } \partial\Omega.$$

If ψ' is not identically zero, then the pressure at $r = R$ will be strictly positive, and so (3.1) does not hold. This example suggests that the approach of section 2 is only practical when one considers symmetrical domains

Ω and symmetrical solutions $\tilde{v} \in H_g(\Omega)$. Condition (3.1) is actually too strong a requirement since \tilde{v} is not just any solution of the Euler equations, but is a certain limit of solutions to the steady Navier-Stokes equations. For definiteness, we shall assume throughout the rest of this paper that Ω is the annulus of Example 3.1 and $f \equiv 0$. The boundary data \tilde{g} is assumed to satisfy (1.4) but we drop the demand of symmetry about $\{x_2 = 0\}$. Recall that we begin by fixing some $v_0 > 0$, let $S(v)$ be as in (1.14), and then try to prove (1.16) by assuming the contrary and deriving a contradiction. Let $v_n \in [v_0, \tilde{v}]$ be such that $|v_n|_H \rightarrow \infty$, where $v_n = u_n - g$. Equations (1.1) - (1.3) lead to certain maximum principles; more precisely, if $\phi_n = p_n + \frac{1}{2} |u_n|^2$ denotes the total head pressure, then

$$v_n \Delta \phi_n - u_n \cdot \nabla \phi_n = v_n \omega_n^2 \text{ in } \Omega, \quad (3.2)$$

where $\omega_n = \frac{\partial}{\partial x_2} (u_n)_1 - \frac{\partial}{\partial x_1} (u_n)_2$ denotes vorticity. Equation (3.2) yields a one-sided maximum principle for ϕ_n : if A is an open subset of Ω , then ϕ_n takes its maximum on \bar{A} at ∂A . This maximum principle was used by Gilbarg and Weinberger [5], [6] to study the difficult problem of steady Navier-Stokes flow past a body in the plane. Define $\tilde{\phi}_n = \phi_n / |v_n|_H^2 = \tilde{p}_n + \frac{1}{2} |\tilde{v}_n + g / |v_n|_H|^2$, and note that $\tilde{\phi}_n$ satisfies a one-sided maximum principle. Since $(\tilde{v}_n, \tilde{p}_n)$ converge in a certain sense to a solution (\tilde{v}, \tilde{p}) of the steady Euler equations, one might expect that the corresponding total head pressure $\tilde{\phi} = \tilde{p} + \frac{1}{2} |\tilde{v}|^2$ satisfies some (weak) version of a one-sided maximum principle. We prove this is true in the following theorem and the remarks thereafter.

THEOREM 3.2. Let s, t denote arbitrary numbers satisfying
 $1 < s < t < R$, and set $A = \{(r, \theta) : s < r < t, 0 < \theta < 2\pi\}$. Then there exist
sequences with $s_j \uparrow s$ and $t_j \downarrow t$ as $j \rightarrow \infty$ such that $\tilde{\phi}(s_j, \cdot)$,
 $\tilde{\phi}(t_j, \cdot) \in C[0, 2\pi]$ and

$$\operatorname{ess\,sup}_{(x,y) \in \Lambda} \tilde{\phi}(x,y) < \max_{\substack{r=s_j, t_j \\ \theta \in [0, 2\pi]}} \tilde{\phi}(r, \theta), \quad j = 1, 2, \dots$$

Proof. For $r \in (1, R)$ define

$$z_n(r) = \int_0^{2\pi} |\tilde{\phi}(r, \theta) - \tilde{\phi}_n(r, \theta)| \left| \frac{\partial}{\partial \theta} \tilde{\phi}(r, \theta) - \frac{\partial}{\partial \theta} \tilde{\phi}_n(r, \theta) \right| d\theta$$

and note that

$$\int_1^R z_n(r) dr < \operatorname{const.} |\tilde{\phi} - \tilde{\phi}_n|_{L^p(\Omega)} |V(\tilde{\phi} - \tilde{\phi}_n)|_{L^q(\Omega)}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $q \in (1, 2)$. Since $\tilde{\phi}, \tilde{\phi}_n$ are bounded in $W^{1,q}(\Omega)$, $q \in (1, 2)$, we may assume that $\tilde{\phi}_n \rightarrow \tilde{\phi}$ in $L^q(\Omega)$ for all $q \in [1, \infty)$. In particular, $z_n \rightarrow 0$ in $L^1(1, R)$ as $n \rightarrow \infty$. Since $z_n \rightarrow 0$ in measure, a suitable subsequence converges to zero uniformly almost everywhere. Given any point $s \in (1, R)$, we can find a sequence $\{s_i\}_{i=1}^\infty$ such that

$$s_i \uparrow s \text{ as } i \rightarrow \infty \text{ and } z_n(s_i) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.3)$$

Define

$$H_n(r) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{\phi}_n(r, \theta) d\theta$$

and similarly for $H(r)$. Since $\tilde{\phi}_n \rightarrow \tilde{\phi}$ in $W^{1,q}(\Omega)$, $q \in (1, 2)$, standard results for traces show that $H_n - H \rightarrow 0$ in $C[1, R]$ as $n \rightarrow \infty$. Let i be fixed and let $\theta_n \in [0, 2\pi]$ be such that

$$\tilde{\phi}_n(s_i, \theta_n) - \tilde{\phi}(s_i, \theta_n) = H_n(s_i) - H(s_i).$$

Now

$$\begin{aligned} & \max_{\theta \in [0, 2\pi]} |\tilde{\phi}_n(s_i, \theta) - \tilde{\phi}(s_i, \theta)|^2 \\ & < |\tilde{\phi}_n(s_i, \theta_n) - \tilde{\phi}(s_i, \theta_n)|^2 + \int_0^{2\pi} \left| \frac{\partial}{\partial \theta} (\tilde{\phi}_n(s_i, \theta) - \tilde{\phi}(s_i, \theta)) \right|^2 d\theta \quad (3.4) \\ & = |H_n(s_i) - H(s_i)|^2 + 2z_n(s_i) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

A similar result holds for a suitable sequence $t_i \uparrow t$. Since $\tilde{\phi}_n$ satisfies a one-sided maximum principle in Ω , we have

$$\hat{\phi}_n(x,y) \leq \max_{\substack{r=s_1, t_1 \\ \theta \in [0, 2\pi]}} \hat{\phi}_n(r, \theta), (x,y) \in A.$$

If we let $n \rightarrow \infty$ and use (3.4), then the theorem is proved. q.e.d.

Remark. A similar version of Theorem 3.2 holds for arbitrary open sets A with $\bar{A} \subset \Omega$. One can find open sets $\{A_j\}_{j=1}^\infty$ with (i) ∂A_j of class C^∞ , (ii) $\bar{A} \subset A_j \subset \bar{A}_j \subset \Omega$, (iii) $\hat{\phi}$ is continuous on ∂A_j , (iv) $\bar{A}_k \subset A_j$ if $k > j$, (v) $A_j \rightarrow A$ in the sense that $A = \text{int} \cap A_j$, and such that

$$\text{ess sup}_A \hat{\phi} \leq \max_{\partial A_j} \hat{\phi}, j = 1, 2, \dots$$

Example 3.1 gives a whole family of solutions to the steady Euler equations in Ω for which the pressure is not the same constant value on $\partial\Omega$. We claim that such flows cannot arise as the limit of unbounded solutions to (1.1) - (1.3) with $f \equiv 0$ and v_0 replaced by v_n . Indeed, let ψ be as in Example 3.1, and note that the total head pressure is only a function of r :

$$\hat{\phi}(r) = \int_1^r (\psi'(w))^2 / w \, dw + \frac{1}{2} |\psi'(r)|^2, r \in [1, R].$$

Since $\psi'' \in L^2(1, R)$ with $\psi'(1) = \psi'(R) = 0$ (which is equivalent to $\hat{v} = (\psi_y, -\psi_x) \in H(\Omega)$) we know that $\hat{\phi}$ is (Hölder) continuous on $[1, R]$. Assume for the moment that $\hat{\phi}$ satisfies Theorem 3.2 and ψ' is not identically zero on $[1, R]$. Then $\hat{\phi}$ takes its maximum on any interval $[s, t]$, with $1 < s < t < R$, at one or both of the endpoints. Since $\hat{\phi}(1) = 0$ and $\hat{\phi} > 0$ on $[1, R]$, it follows that $\hat{\phi}$ is non-decreasing on $[1, R]$. A calculation gives

$$\hat{\phi}(t) - \hat{\phi}(s) = \int_s^t \frac{\psi'(w)}{w} (w\psi'(w))' \, dw,$$

for any $s, t \in [1, R]$ whence

$$0 < \psi'(w)(w\psi'(w))' \text{ almost everywhere on } [1, R]. \quad (3.5)$$

Since ψ' is not identically zero on $[1, R]$ by hypothesis, there exists some $w_0 \in (1, R)$ such that $\psi'(w_0) \neq 0$. Let B denote the largest open

interval containing v_0 such that $\psi' \neq 0$ on B . Since $\psi'(1) = \psi'(R) = 0$, we know that ψ' vanishes at the endpoints of B . On the other hand, (3.5) says that $(\psi'(w))'$ is one signed (almost everywhere) on B . The only possibility is that $(\psi'(w))' = 0$ on B , whence $\psi' = 0$ on B . This is a contradiction, and so the non-trivial solutions (\hat{v}, \hat{p}) of Example 3.1 do not satisfy Theorem 3.2. Therefore, it is natural to ask the follow question:

Let $\Omega \subset \mathbb{R}^2$ denote an annulus and let (\hat{v}, \hat{p}) be a weak solution of the Euler equations $(\hat{v} \cdot \nabla) \hat{v} = -\nabla \hat{p}$ in Ω with $\hat{v} \in H(\Omega)$. If $\hat{\phi} = \hat{p} + \frac{1}{2} |\hat{v}|^2$ satisfies Theorem 3.2 and the remark thereafter, then does $\hat{p}|_{r=1} = \hat{p}|_{r=R}$, or, equivalently, does $\hat{\phi}|_{r=1} = \hat{\phi}|_{r=R}$?

There are other maximum principles associated with (1.1) - (1.3) with v_0 replaced by v and $f \equiv 0$. If we denote the vorticity by

$$\omega = \frac{\partial}{\partial x_2} u_1 - \frac{\partial}{\partial x_1} u_2, \text{ then a calculation gives}$$

$$-v \Delta \omega + u \cdot \nabla \omega = 0 \text{ in } \Omega,$$

so that ω satisfies a two-sided maximum principle. If $\{u_n\}_{n=1}^\infty$ are solutions of (1.1) - (1.3) with v_0 replaced by v_n , then

$$\hat{\omega}_n = \frac{1}{|\hat{v}_n|_H} \left\{ \frac{\partial}{\partial x_2} (u_n)_1 - \frac{\partial}{\partial x_1} (u_n)_2 \right\}$$

satisfies a two-sided maximum principle in Ω : if A is an open subset of Ω , then the maximum and minimum values of $\hat{\omega}_n$ on \bar{A} occur on ∂A . Although $\hat{\omega}_n$ is bounded in $L^2(\Omega)$ and converges weakly to $\hat{\omega} = \frac{\partial}{\partial x_2} \hat{v}_1 - \frac{\partial}{\partial x_1} \hat{v}_2$, this is not enough to prove results analogous to Theorem 3.2 for $\hat{\omega}$; we would need some control on derivatives of $\hat{\omega}_n$.

If (\hat{v}, \hat{p}) is a weak solution to the steady Euler equations in Ω with $\hat{v} \in H(\Omega)$ (and \hat{v} is not necessarily a limit of Navier-Stokes solutions), then the total head pressure $\hat{\phi} \in W^{1,q}(\Omega)$, $q \in (1,2)$, whence $\hat{\phi} \in L^q(\Omega)$ for all $q \in [1, \infty)$. By using the averaging methods of Gilbarg and Weinberger [5],

[6] for the pressure \hat{p} , one can show that $\hat{p} \in C(\bar{\Omega})$ and $\hat{v}(x,y) \rightarrow 0$ as $(x,y) \rightarrow \partial\Omega$. If we know in addition that Theorem 3.2 and the remark after it hold, then we have $\hat{\theta} \in L^{\infty}(\Omega)$. It is not known if $\hat{\theta} \in C(\bar{\Omega})$.

REFERENCES

1. CATTABRIGA, L., Si un problema al contorno relativo al sistema di equazioni di Stokes, Rend. Sem. Mat. Padova 31 (1961), 1-33.
2. FINN, R., On the steady-state solutions of the Navier-Stokes equations, III, Acta Math. 105 (1961), 197-244.
3. FOIAS, C. and TEMAM, R., Structure of the set of stationary solutions of the Navier-Stokes equations, Comm. Pure Appl. Math. 30 (1977), 149-164.
4. FOIAS, C. and TEMAM, R., Remarques sur les équations de Navier-Stokes stationnaires et les phénomènes de bifurcation, Ann. Scuola Norm. Sup. Pisa (4) 5 (1978), 29-63.
5. GILBARG, D. and WEINBERGER, H. F., Asymptotic properties of Leray's solution of the stationary two-dimensional Navier-Stokes equations, Russ. Math. Surv. 29 (1974), 109-123.
6. GILBARG, D. and WEINBERGER, H. F., Asymptotic properties of steady plane solutions of the Navier-Stokes equations with bounded Dirichlet integral, Ann. Scuola Norm. Sup. Pisa (4) 5 (1978), 381-404.
7. HEYWOOD, J. G., The Navier-Stokes equations: on the existence, regularity and decay of solutions, Indiana Univ. Math. J. 29 (1980), 639-681.
8. LADYZHENSKAYA, O. A., The mathematical theory of viscous incompressible flow. Second edition, Gordon and Breach, 1969.
9. LERAY, J., Etude de diverses équations intégrales nonlinéaires et de quelques problèmes que pose l'hydrodynamique, J. Math. Pures Appl. 12 (1933), 1-82.

10. SAUT, J. C. and TEMAM, R., Propriétés de l'ensemble des solutions stationnaires ou périodiques des équations de Navier-Stokes généralisées par rapport aux données aux limites, C. R. Acad. Sci. Paris 285 (1977), 673-676.
11. TEMAM, R., Qualitative properties of Navier-Stokes equations, Communication at the International Symposium on Partial Differential Equations and Continuum Mechanics, Rio de Janeiro 1977, Proceedings edited by L. A. Madeiros, North-Holland 1978.
12. TEMAM, R., Navier-Stokes equations: theory and numerical analysis. North-Holland, 1979.

CJA/jvs

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER #2509	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Existence of Solutions to the Nonhomogeneous Steady Navier-Stokes Equations		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Charles J. Amick		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P. O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE April 1983
		13. NUMBER OF PAGES 20
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Navier-Stokes equations, a priori bounds, maximum principles		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) This paper concerns the existence of steady solutions to the Navier-Stokes equations in a bounded domain $\Omega \subset \mathbb{R}^2$. The condition of solenoidality for the velocity field imposes a necessary condition on the boundary data. For a certain class of symmetrical domains, we show that this necessary condition implies the existence of a solution to the problem. The method consists of proving a priori bounds on solutions by assuming the contrary, rescaling the equations, and then arriving at a solution to the steady Euler equations in the limit. Examination of this equation leads to the desired contradiction. After		

ABSTRACT (continued)

one has suitable bounds on any solutions, one uses the Leray-Schauder theorem to prove existence.

In addition, we remark on the problem of a general bounded domain $\Omega \subset \mathbb{R}^2$, and suggest how certain maximum principles might yield the expected results.

